5 Chapter 5

5.1 Q10

Assume \( p > 0 \), \( \int_E |f - f_k|^p \to 0 \), and \( \int_E |f_k|^p \leq M \) for all \( k \).

By Q9, there is a subsequence \( f_{k_j} \to f \) a.e. in \( E \). Since \( t \mapsto |t|^p \) is continuous, \( |f_{k_j}|^p \to |f|^p \) a.e. in \( E \).

By Fatou’s Lemma,

\[
\int_E |f|^p \leq \liminf_{j \to \infty} \int_E |f_{k_j}|^p \leq M.
\]

5.2 Q14

Let \( f \in L^p(E) \). WLOG we may assume \( f \geq 0 \) since for \( a > 0 \),

\[
0 \leq a^p \omega(a) = a^p |\{ f > a \}| \leq a^p |\{|f| > a\}|.
\]

Thus if \( \lim_{a \to 0^+} a^p |\{|f| > a\}| = 0 \), that would imply \( \lim_{a \to 0^+} a^p \omega(a) = 0 \).

Let \( \epsilon > 0 \).

Lemma 5.1. We may choose \( \delta > 0 \) such that \( \int_{\{f \leq \delta\}} f^p < \epsilon \).
Proof. Define 

\[ f_k(x) = \begin{cases} 
0 & \text{if } 0 \leq f(x) \leq \frac{1}{k} \\
 f(x) & \text{otherwise.} 
\end{cases} \]

Then \( f_k^p \to f^p \) and \( |f_k^p| \leq f^p \). Since \( f^p \in L(E) \), by Lebesgue’s DCT, \( \int_E f_k^p \to \int_E f^p \). There exists \( K \) such that for \( k \geq K \), \( |\int_E f^p - \int_E f_k^p| = |\int_{\{f \leq \frac{1}{k}\}} f^p| < \epsilon \).

Take \( \delta = \frac{1}{K} \), then \( \int_{\{f \leq \delta\}} f^p < \epsilon \). \( \square \)

Note that \( \omega(a), \omega(\delta) < \infty \) since \( f \in L^p(E) \). Thus
\[
\begin{align*}
& a^p [\omega(a) - \omega(\delta)] = a^p |\{f > a\} - |\{f > \delta\}| \\
& = a^p |\{a < f \leq \delta\}| \\
& = \int_{\{a < f \leq \delta\}} a^p \\
& \leq \int_{\{a < f \leq \delta\}} f^p \\
& < \epsilon
\end{align*}
\]

for \( 0 < a < \delta \).

Rearranging, we get \( a^p \omega(a) < \epsilon + a^p \omega(\delta) \). Letting \( a \to 0^+ \) gives
\[
\lim_{a \to 0^+} a^p \omega(a) = 0
\]

since \( \epsilon > 0 \) is arbitrary.

5.3 Q15

Since \( \omega(\alpha) \) is a decreasing function, for \( a > 0 \) we have

\[
\int_{a/2}^{a} \alpha^{p-1} \omega(\alpha) \, d\alpha \geq \omega(a) \int_{a/2}^{a} \alpha^{p-1} \, d\alpha \\
= \omega(a) \left[ \frac{\alpha^p}{p} \right]_{a/2}^{a} \\
= \omega(a) a^p \left( \frac{2^{p} - 1}{2^p} \right).
\]
Thus,
\[ a^p \omega(a) \leq \frac{2^p - 1}{2^p} \int_{a/2}^{a} a^{p-1} \omega(\alpha) \, d\alpha \leq \frac{2^p - 1}{2^p} \int_0^a a^{p-1} \omega(\alpha) \, d\alpha. \]

**Lemma 5.2.** \( \lim_{a \to 0^+} \int_0^a a^{p-1} \omega(\alpha) \, d\alpha = 0. \)

**Proof.** For \( 0 < a < 1 \), we have
\[ \int_0^a a^{p-1} \omega(\alpha) \, d\alpha = \int_0^1 a^{p-1} \omega(\alpha) \, d\alpha - \int_a^1 a^{p-1} \omega(\alpha) \, d\alpha. \]

Note that \( \int_a^1 a^{p-1} \omega(\alpha) \, d\alpha = \int_0^1 a^{p-1} \omega(\alpha) \cdot \chi_{[a,1]} \, d\alpha. \) Let \( a_k \to 0^+ \), then note that
\[ 0 \leq a^{p-1} \omega(\alpha) \chi_{[a_k,1]} \not
\]}

on \((0,1)\) thus by Monotone Convergence Theorem,
\[ \int_a^1 a^{p-1} \omega(\alpha) \, d\alpha \to \int_0^1 a^{p-1} \omega(\alpha) \, d\alpha \]
as \( a \to 0^+ \). This proves \( \lim_{a \to 0^+} \int_0^a a^{p-1} \omega(\alpha) \, d\alpha = 0. \)

Hence \( \lim_{a \to 0^+} a^p \omega(a) = 0 \) as a direct consequence of the lemma.

Similarly for \( b > 0 \) we have
\[ b^p \omega(b) \leq \frac{2^p - 1}{2^p} \int_{b/2}^b a^{p-1} \omega(\alpha) \, d\alpha. \]

**Lemma 5.3.** \( \lim_{b \to \infty} \int_{b/2}^b a^{p-1} \omega(\alpha) \, d\alpha = 0. \)

**Proof.** Write
\[ \int_{b/2}^b a^{p-1} \omega(\alpha) \, d\alpha = \int_0^b a^{p-1} \omega(\alpha) \, d\alpha - \int_0^{b/2} a^{p-1} \omega(\alpha) \, d\alpha. \]

By similar argument using Monotone Convergence Theorem, we have
\[ \lim_{b \to \infty} \int_0^b a^{p-1} \omega(\alpha) \, d\alpha = \lim_{b \to \infty} \int_0^{b/2} a^{p-1} \omega(\alpha) \, d\alpha = \int_0^\infty a^{p-1} \omega(\alpha) \, d\alpha < \infty. \]

Thus \( \lim_{b \to \infty} \int_{b/2}^b a^{p-1} \omega(\alpha) \, d\alpha = 0. \)

This proves \( \lim_{b \to \infty} b^p \omega(b) = 0. \)
5.4 Q16

Let \( E_{ab} = \{ x \in E : a < f(x) \leq b \} \) for \( 0 < a < b < \infty \). We quote a theorem from the textbook:

**Theorem** (Theorem 5.46). If \( a < f \leq b \) (\( a \) and \( b \) finite) in \( E \) (\(|E| < \infty \)) and \( \phi \) is continuous on \([a,b]\), then \( \int_E \phi(f) = -\int_a^b \phi(\alpha) \, d\omega(\alpha) \).

Note that \(|E_{ab}| \leq \omega(a) < \infty \) and \( \phi(\alpha) = \alpha^p \) is continuous. Applying Theorem 5.46, we have
\[
\int_{E_{ab}} f^p = -\int_a^b \alpha^p \, d\omega(\alpha).
\]

Taking limits as \( a \to 0^+ \) and \( b \to \infty \), we get
\[
\int_E f^p = -\int_0^\infty \alpha^p \, d\omega(\alpha)
\]
by Monotone Convergence Theorem, since \( f^p \chi_{E_{ab}} \to f^p \) on \( E \).

If \( \int_0^\infty \alpha^p \, d\omega(\alpha) = -\infty \) and \( \int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha = \infty \), then Theorem 5.51 holds since
\[
\infty = \int_E f^p = -\int_0^\infty \alpha^p \, d\omega(\alpha) = p \int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha.
\]

Next, assume either \( \int_0^\infty \alpha^p \, d\omega(\alpha) \) or \( \int_0^\infty \alpha^{p-1} \omega(\alpha) \) is finite.

By Theorem 2.21 (integration by parts), if \( 0 < a < b < \infty \), we have
\[
-\int_a^b \alpha^p \, d\omega(\alpha) = -b^p \omega(b) + a^p \omega(a) + p \int_a^b \alpha^{p-1} \omega(\alpha) \, d\alpha \tag{5.1}
\]
using the fact that \( \alpha^p \) is continuously differentiable on \([a,b]\).

**Case 1** If \( \int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha < \infty \), using Q15 and taking limits as \( a \to 0^+ \), \( b \to \infty \) in (5.1), we get
\[
-\int_0^\infty \alpha^p \, d\omega(\alpha) = 0 + 0 + p \int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha.
\]
Case 2) If $|\int_0^\infty \alpha^p \, d\omega(\alpha)| < \infty$, then $\int_E f^p = -\int_0^\infty \alpha^p \, d\omega(\alpha) < \infty$, i.e. $f \in L^p(E)$. Thus Lemma 5.50 and Q14 holds so that $\lim_{b \to \infty} b^p \omega(b) = \lim_{a \to 0^+} a^p \omega(a) = 0$. Hence taking limits as $a \to 0^+$, $b \to \infty$ in (5.1), we get

$$-\int_0^\infty \alpha^p \, d\omega(\alpha) = p \int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha.$$

5.5 Q18

Let $f \geq 0$. By Question 16,

$$\int_E f^p = p \int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha,$$

thus $f \in L^p$ iff $\int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha < \infty$.

The key observation is

$$\int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha = \sum_{k=-\infty}^{\infty} \int_{2k}^{2k+1} \alpha^{p-1} \omega(\alpha) \, d\alpha. \quad (5.2)$$

Since $\alpha^{p-1}$ is increasing and $\omega(\alpha)$ is decreasing, we have

$$\int_{2k}^{2k+1} (2k)^{p-1} \omega(2^{k+1}) \leq \int_{2k}^{2k+1} \alpha^{p-1} \omega(\alpha) \leq \int_{2k}^{2k+1} (2^{k+1})^{p-1} \omega(2^k).$$

Simplifying, we get

$$2^{-p}[2^{(k+1)p} \omega(2^{k+1})] \leq \int_{2k}^{2k+1} \alpha^{p-1} \omega(\alpha) \leq 2^{-p-1}[2^{kp} \omega(2^k)].$$

Summing from $k = -\infty$ to $k = \infty$, we get

$$2^{-p} \sum_{k=-\infty}^{\infty} 2^{(k+1)p} \omega(2^{k+1}) \leq \int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha \leq 2^{-p-1} \sum_{k=-\infty}^{\infty} 2^{kp} \omega(2^k).$$

Note that the left most term

$$2^{-p} \sum_{k=-\infty}^{\infty} 2^{(k+1)p} \omega(2^{k+1}) = 2^{-p} \sum_{k=-\infty}^{\infty} 2^{kp} \omega(2^k)$$

by change of index in summation.

Therefore $\sum_{k=-\infty}^{\infty} 2^{kp} \omega(2^k) < \infty$ iff $\int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha < \infty$ iff $f \in L^p$. 5
We first prove the statement for any indicator function $\chi_{E_1}$, where $E_1 \subseteq E$ is measurable. We will use the following theorem.

**Theorem** (Theorem 3.35). Let $T$ be a linear transformation of $\mathbb{R}^n$, and let $E$ be measurable. Then $|TE| = |\det T||E|$.

We have
\[
|\det T| \int_{T^{-1}E} \chi_{E_1}(Tx) \, dx = |\det T| \int_{T^{-1}E} \chi_{T^{-1}E_1}(x) \, dx \\
= |\det T||T^{-1}E_1| \\
= |\det T||\det T^{-1}||E_1| \quad \text{(by Theorem 3.35)} \\
= |E_1| \quad \text{(since } |\det T||\det T^{-1}| = 1) \\
= \int_E \chi_{E_1}(y) \, dy.
\]

By linearity of the integral, the statement is also true for any simple function $f(x) = \sum_{k=1}^{N} a_k \chi_{E_k}(x)$, where $E_1, \ldots, E_N$ are measurable.

Write $f = f^+ - f^-$. Since $f^+ \geq 0$, there is an increasing sequence of measurable simple functions $f_k \uparrow f^+$. Then by Monotone Convergence Theorem,
\[
\int_E f^+(y) \, dy = |\det T| \int_{T^{-1}E} f^+(Tx) \, dx.
\]

Since $f^- \geq 0$, similarly the statement is also true for $f^-$. Since $\int_E f(y) \, dy$ exists, at least one of the integrals $\int_E f^+(y) \, dy$, $\int_E f^-(y) \, dy$ is finite (so the case $\infty - \infty$ will not occur), thus we may conclude that
\[
\int_E f(y) \, dy = \int_E f^+(y) \, dy - \int_E f^-(y) \, dy \\
= |\det T| \int_{T^{-1}E} f^+(Tx) \, dx - |\det T| \int_{T^{-1}E} f^-(Tx) \, dx \\
= |\det T| \int_{T^{-1}E} f(Tx) \, dx.
\]
5.7 Q21

We will use the following theorem:

**Theorem** (Theorem 5.11). Let \( f \) be nonnegative and measurable on \( E \). Then \( \int_E f = 0 \) if and only if \( f = 0 \) a.e. in \( E \).

We have \( \int_{\{f \geq 0\}} f = 0 \) since \( \{f \geq 0\} \) is a measurable subset of \( E \). Thus by Theorem 5.11, \( f = 0 \) a.e. in \( \{f \geq 0\} \).

Next we have \( \int_{\{f < 0\}} f = 0 \) since \( \{f < 0\} \) is a measurable set. This implies \( \int_{\{f < 0\}} (-f) = -0 = 0 \). Since \( -f \) is nonnegative and measurable on \( \{f < 0\} \), this implies \( -f = 0 \) a.e. in \( \{f < 0\} \).

Therefore \( f = 0 \) a.e. in \( E = \{f \geq 0\} \cup \{f < 0\} \).

6 Chapter 6

6.1 Q2

Let \( F(x, y) := f(x) \), \( G(x, y) := g(y) \) for all \( x, y \in \mathbb{R}^n \). Observe that

\[
\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n | F(x, y) > \alpha\} = \{x \in \mathbb{R}^n | f(x) > \alpha\} \times \mathbb{R}^n
\]

which is measurable by repeated application of Lemma 5.2 which states that \( E \times \mathbb{R} \) is measurable for measurable \( E \subseteq \mathbb{R}^m \). Thus, \( F(x, y) \) is measurable in \( \mathbb{R}^n \times \mathbb{R}^n \).

Similarly,

\[
\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n | G(x, y) > \alpha\} = \mathbb{R}^n \times \{y \in \mathbb{R}^n | g(y) > \alpha\}
\]

is a measurable set. Thus, \( G(x, y) \) is measurable in \( \mathbb{R}^n \times \mathbb{R}^n \).

Hence \( F(x, y)G(x, y) = f(x)g(y) \) is measurable in \( \mathbb{R}^n \times \mathbb{R}^n \).

Let \( E_1 \) and \( E_2 \) be measurable subsets of \( \mathbb{R}^n \). Then \( \chi_{E_1} \) and \( \chi_{E_2} \) are measurable in \( \mathbb{R}^n \). By the earlier part, \( \chi_{E_1}(x) \chi_{E_2}(y) \) is measurable in \( \mathbb{R}^n \times \mathbb{R}^n \).
Note that $\chi_{E_1}(x)\chi_{E_2}(y) = \chi_{E_1 \times E_2}(x, y)$, so $E_1 \times E_2$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.

$$|E_1 \times E_2| = \int_{\mathbb{R}^n \times \mathbb{R}^n} \chi_{E_1 \times E_2}(x, y) \, dx \, dy = \int_{\mathbb{R}^n \times \mathbb{R}^n} \chi_{E_1}(x)\chi_{E_2}(y) \, dx \, dy = \int_{\mathbb{R}^n} \chi_{E_1}(x) \left[ \int_{\mathbb{R}^n} \chi_{E_2}(y) \, dy \right] \, dx \quad \text{(by Tonelli’s Theorem)}$$

$$= \int_{\mathbb{R}^n} \chi_{E_1}(x) \, dx \cdot |E_2| = |E_1||E_2|.$$

6.2 Q4

By Lemma 6.15, $f(x+t)$ and $f(-x+t)$ are both measurable in $\mathbb{R}^2$. By Tonelli’s Theorem,

$$\int\int_{[0,1]^2} |f(x+t) - f(-x+t)| \, dt \, dx = \int_0^1 \left[ \int_0^1 |f(x+t) - f(-x+t)| \, dt \right] \, dx$$

$$\leq \int_0^1 c \, dx$$

$$= c.$$

We quote the following result:

**Theorem** (Chapter 5 Exercise 20). Let $y = Tx$ be a nonsingular linear transformation of $\mathbb{R}^n$. If $\int_E f(y) \, dy$ exists, then

$$\int_E f(y) \, dy = |\det T| \int_{T^{-1}E} f(Tx) \, dx.$$
Let \( T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \) so that \( \begin{pmatrix} \xi \\ \eta \end{pmatrix} = T \begin{pmatrix} x \\ t \end{pmatrix} \). Let \( E = [0, 1]^2 \). We compute that \( T^{-1}E \) is the convex hull of \( \{(0, 0), (-\frac{1}{2}, \frac{1}{2}), (0, 1), (\frac{1}{2}, \frac{1}{2})\} \).

\[
\iint_{[0,1]^2} |f(\xi) - f(\eta)| \, d\eta \, d\xi \\
= |\det T| \iint_{T^{-1}E} |f(x + t) - f(-x + t)| \, dt \, dx \\
\leq 2 \iint_{[-1,1] \times [0,1]} |f(x + t) - f(-x + t)| \, dt \, dx \quad \text{(since } T^{-1}E \subseteq [-1, 1] \times [0, 1])
\]

\[
= 4 \iint_{[0,1]^2} |f(x + t) - f(-x + t)| \, dt \, dx \quad \text{(by periodicity of } f) \\
\leq 4c.
\]

Hence \( f(\xi) - f(\eta) \) is integrable over the square \([0, 1]^2\). By Q3, we conclude that \( f \in L[0, 1] \).

**6.3 Q5**

(a) \( \int_E f = |R(f, E)| \) \quad \text{(by definition of the integral)}

\[
= \iint_{R(f, E)} dx \, dy \\
= \int_0^\infty \left[ \int_{\{x : (x, y) \in R(f, E)\}} dx \right] dy \quad \text{(by Tonelli’s Theorem)}
\]

\[
= \int_0^\infty \{|x \in E : f(x) \geq y\} \, dy \\
= \int_0^\infty \omega(y) \, dy.
\]
The last equality follows from the fact that $\omega(y)$ is decreasing thus has countably many points of discontinuity, and $\omega(y) = |\{x \in E : f(x) \geq y\}|$ unless $y$ is a point of discontinuity of $\omega$.

(b)

Note that $f^p(x) = \int_0^{f(x)} py^{p-1} dy$ for all $x \in E$. Thus

$$
\int_E f^p(x) \, dx = \int_E \int_0^{f(x)} py^{p-1} \, dy \, dx
$$

$$
= \int_{\mathbb{R}(f,E)} py^{p-1} \, dy \, dx \quad \text{(by Tonelli’s Theorem)}
$$

$$
= \int_0^{\infty} \int_{\{x \in E : f(x) \geq y\}} py^{p-1} \, dx \, dy \quad \text{(by Tonelli’s Theorem)}
$$

$$
= p \int_0^{\infty} y^{p-1} \omega(y) \, dy
$$

since $\omega(y) = |\{x \in E : f(x) \geq y\}|$ almost everywhere.